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Bounded geodesics in rank-1 locally symmetric spaces

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Abstract. Let M be a rank 1 locally symmetric space of finite Riemannian volume. It is proved that the set of unit vectors on a non-constant C^1 curve in the unit tangent sphere at a point $p \in M$ for which the corresponding geodesic is bounded (relatively compact) in M, is a set of Hausdorff dimension 1.

1. *Introduction*

Let M be a rank-1 locally symmetric space of non-compact type with finite Riemannian volume. The geodesic flow on the unit tangent bundle SM of M is known to be ergodic (cf [**Ma**], [**Mo**]) and it follows that for almost all $(p, v) \in SM$, where $p \in M$ and v is a unit tangent vector at p, the geodesic through p in the direction of v is dense in M. A theorem from Dani (cf [**D**, Theorem 5.1]) says that the set C of all $(p, v) \in SM$ for which the corresponding geodesic is bounded (namely those with compact closure in M), though of measure zero, is 'large' in the sense that its Hausdorff dimension is equal to that of the unit tangent bundle itself. This theorem was strengthened in [**A**] for the case of constant negative curvature wherein it was proved that the set of unit tangent vectors positioned at some point $p \in M$, lying on any non-constant C^1 curve in the unit sphere S_p and determining bounded geodesics from p is of Hausdorff dimension 1. This raises the question about whether a similar strengthening holds in the case of rank-1 locally symmetric spaces as considered by Dani. In this paper we answer the question in the affirmative. That is, we prove:

THEOREM 1.1. *Let M be a rank-1 locally symmetric space of non-compact type with finite Riemannian volume. Let $p \in M$ and S_p be the unit tangent sphere at p. Let C_p be the subset of S_p consisting of all those vectors v such that the geodesic starting from p in the direction of v is a bounded subset of M. Let $\sigma: [0, 1] \to S_p$ be any non-constant, C^1 curve. Then, $\sigma \cap C_p$ is of Hausdorff dimension 1 with respect to the metric on S_p restricted to σ.*

The proof given here closely follows the geometric argument given in [**A**] for the case of constant negative curvature but involves the trigonometric formulae for rank-1 symmetric spaces of non-compact type, developed in the Appendix. It is worth mentioning the following corollary on the dynamics of the geodesic flow on SM where SM is equipped with the usual Riemannian metric; in [**A**] this could be proved only in the constant curvature case.

COROLLARY 1.2. *The set C of $(p, v) \in SM$ for which the corresponding geodesic is bounded has Hausdorff dimension equal to $2n-1$, where $n = \dim M$*

Following [**D**], we adapt Schmidt's strategy [**S**], and prove that the set $\sigma \cap C_p$ as in Theorem 1.1 contains 'α-winning set' of what is now called Schmidt game. For readers' convenience let us describe the game here. The game involves two players, say, $\mathcal{A}$ and $\mathcal{B}$, a complete metric space X and two numbers $\alpha, \beta \in (0, 1) = \{t \in \mathbb{R}: 0 < t < 1\}$. The game begins with $\mathcal{B}$ choosing a closed ball B_0 in X with arbitrary positive radius. Then $\mathcal{A}$ chooses a closed ball A_1 contained in B_0 with radius α times the radius of B_0. Next $\mathcal{B}$ chooses a closed ball B_1 contained in A_1 of radius β times that of A_1 and so on; the game thus proceeds inductively by $\mathcal{A}$ choosing a closed ball A_k in B_{k-1} with radius α times that of B_{k-1} and then $\mathcal{B}$ choosing a closed ball B_k contained in A_k having radius β times that of A_k. The closed balls A_k, and hence B_k, intersect in a unique point since X is complete. A subset S of X is called an (α, β)-winning set for $\mathcal{A}$ if $\mathcal{A}$ can play in such a way that the unique intersection point always lies in S, whatever the choices of $\mathcal{B}$ are during his turns. An α-winning set is an (α, β)-winning set for all $\beta \in (0, 1)$. Theorem 1.1 will be proved by showing that for some subinterval I, $\sigma(I) \cap C_p$ is an α-winning set for all $\alpha \in (0, 1/2)$. By a result of Schmidt [**S**, Corollary 2, §11] this implies the assertion as in the theorem.

2. *Proof of Theorem 1.1*

We can assume, after normalization, that the curvature of M is pinched between -4 and -1. Let $\tilde{M}$ denote the simply connected cover of M and let $\pi: \tilde{M} \to M$ be the covering projection map. Let $\tilde{M}(\infty)$ denote the sphere at infinity which is the set of asymptotic classes of geodesic rays in $\tilde{M}$. The fundamental group Γ of M acts as a discrete group of isometries on $\tilde{M}$ and its action extends continuously to $\tilde{M}(\infty)$. It is shown in [**E**, Theorem 3.1] that M, as above, has only finitely many ends and that each end $\mathcal{E}$ has a neighborhood U which is the projection under π of a precisely invariant horoball $\tilde{U}$ based at a point $x \in \tilde{M}(\infty)$ corresponding to the end $\mathcal{E}$. This means that for $\phi \in \Gamma$, $\phi(\tilde{U}) = \tilde{U}$ whenever $\phi(x) = x$ and $\phi(\tilde{U}) \cap \tilde{U} = \emptyset$ whenever $\phi(x) \neq x$.

Now let $\tilde{p} \in \pi^{-1}(p) \subset \tilde{M}$ and let $S_{\tilde{p}}$ denote the space of unit vectors at $\tilde{p}$ and $\tilde{\sigma} \subset S_{\tilde{p}}$ be a lift of the curve σ and $C_{\tilde{p}}$ be the lift of C_p. Since $\tilde{\sigma}$ is a non-constant C^1 curve, it contains an embedded, regular (i.e. nonzero derivative at each point) C^1 segment and on such a segment the two distances, the time parameter distance d and the distance d' as a subset of $S_{\tilde{p}}$, are bi-lipschitz related with respect to each other. That is, for this regular segment of $\tilde{\sigma}$ there exists a constant $\delta \geq 1$ such that,

$$\delta^{-1} d(\tilde{\sigma}(s), \tilde{\sigma}(t)) \leq \delta d'(\tilde{\sigma}(s), \tilde{\sigma}(t)) \leq \delta d(\tilde{\sigma}(s), \tilde{\sigma}(t)). \tag{2.1}$$

In proving the theorem there is no loss of generality in replacing $\tilde{\sigma}$ by the segment as above. Further, since the distances d and d' on this segment are bi-lipschitz related and since bi-lipschitz maps preserve Hausdorff dimension, it is enough to prove the theorem with respect to the distance d. This enables us to use the following lemma of Schmidt (cf. [S, Lemma 15]) later in this paper.

LEMMA 2.1. *Give $\alpha, \beta \in (0, 1)$ such that $1 - 2\alpha + \alpha\beta > 0$, let $\nu = \frac{1}{2}(1 - 2\alpha + \alpha\beta)$ and let h be a positive integer such that $(\alpha\beta)^h < \nu$. Consider an (α, β)-game on the real line $\mathbb{R}$. Let B_k be the closed ball with centre b_k and radius ρ_k chosen by $\mathcal{B}$ at the kth stage of the game. Let J be an interval with center c and radius ρ such that*

$$\rho \leq |c - b_k| + \nu\rho_k. \tag{2.2}$$

Then, $\mathcal{A}$ can make his choices in such a way that B_{k+h} is disjoint from J.

Now let $\mathcal{E}$ be any end of M. Let $x_0 \in \tilde{M}(\infty)$ be a point corresponding to the end $\mathcal{E}$. Now, there exists a neighborhood of the end $\mathcal{E}$ such that its preimage in $\tilde{M}$ consists of a set of pairwise disjoint horoballs at distinct Γ-translates of x_0. Let us fix one such set of horoballs $\mathcal{U}$ corresponding to a neighborhood $U = F(N \times (0, \infty))$ of the end $\mathcal{E}$. Also, for $s > 0$, we denote by $\mathcal{U}_s$ the set of horoballs corresponding to the neighborhood $U_s = F(N \times (s, \infty))$ and refer to $\mathcal{U}_s$ as shrinking of $\mathcal{U}$ by a factor s. Denote by $C_p(\mathcal{E})$, the set of those vectors for which the corresponding geodesics are bounded away from the end $\mathcal{E}$ and by $C_{\tilde{p}}(\mathcal{E})$, its lift to $S_{\tilde{p}}$. We shall prove that $\tilde{\sigma} \cap C_{\tilde{p}}(\mathcal{E})$ is an (α, β)-winning set for α, β such that $1 - 2\alpha + \alpha\beta > 0$. This is achieved in three steps; for convenience in the proofs of these we assume certain trigonometric formulae for rank-1 symmetric spaces which will be proved separately later in the Appendix.

Step 1. We find $s' > 0$, and a decomposition of $\mathcal{U}_{s'}$ into subcollections $\mathcal{U}_{s'}^k$, for each nonnegative integer k, such that the shadow of the ball B_{kh}, where h is as in Lemma 2.1, chosen by $\mathcal{B}$ at the khth stage of the game, intersects at most one member of the subcollection $\mathcal{U}_{s'}^k$. Here, 'shadow' of a set $X \subseteq S_{\tilde{p}}$ means the union of all geodesics $\{\gamma(t) : t \geq 0, \gamma(0) = \tilde{p} \quad \text{and} \quad \gamma'(0) \in X\}$.

Denote by ρ_0 the radius of the ball B_0 chosen by $\mathcal{B}$. Put $\varepsilon_k = 2\delta^2(\alpha\beta)^{kh}\rho_0 = 2\delta^2\rho_{kh} > 0$. Define r_k by the equation

$$\cosh(r_k)\sin(\varepsilon_k/2) = 1. \tag{2.3}$$

Then any geodesic segment which is at a distance bigger than r_k from $\tilde{p}$ subtends an angle less than or equal to ε_k at $\tilde{p}$. In fact, by Proposition A.3(ii) the subtended angle at distance $r \geq r_k$ is $2\lambda(A, \mu)$ with

$$\sin(\lambda(A, \mu)) \leq \frac{1}{\cosh r} \leq \frac{1}{\cosh(r_k)} = \sin(\varepsilon_k/2)$$

hence $\lambda(A, \mu) \leq \varepsilon_k/2$.

Consider a geodesic triangle Δ_k in $\tilde{M}$ with sides of length a_k, b_k, c_k and a Riemannian angle ε_k opposite to the side c_k, where $a_k, b_k \in [r_{k-1}, r_k + 1]$. We now prove that for any such geodesic triangle and any k, c_k is bounded above by some constant independent of k. We have the following two cases:

Case (a). Each of the other two angles of Δ_k are at most $\pi/2$.

In this case, since $\tilde{M}$ is a Hadamard manifold, c_k is clearly bounded above by the length t_k of the third side of an isosceles triangle whose two sides are of length (r_k+1) each and making an angle ε_k between them. Therefore, using cosine formula A.1(iii) we have,

$$\begin{aligned}\cosh^2 c_k < \cosh^2 t_k &= (\cosh^2(r_k+1) - \sinh^2(r_k+1)\cos(\varepsilon_k))^2 \\ &\quad +(\sinh^2(r_k+1)\cos(\theta(C)))^2 \\ &= (1+2(\sinh^2 r_k + 1)\sin^2(\varepsilon_k/2))^2 \\ &\quad +\sinh^4(r_k+1)\cos^2(\theta(C)) \\ &< (1+8\cosh^2(1))^2 + 16\cosh^4(1) \\ &= C_1^2 \text{ (say)}.\end{aligned}$$

The second inequality above is obtained by using (2.3) and A.3(i).

Case (b). One of the other two angles, say ϕ, opposite to the side a_k, is bigger than $\pi/2$.

Put $\phi = \pi/2 + \psi$. Again, by the cosine formula A.1(iii), we have,

$$\begin{aligned}\cosh^2 a_k &\geq (\cosh b_k \cosh c_k - \sinh b_k \sinh c_k \cos(\pi/2+\psi))^2 \\ &= (\cosh b_k \cosh c_k + \sinh b_k \sinh c_k \sin(\psi))^2 \\ &> \cosh^2 b_k \cosh^2 c_k.\end{aligned}$$

Therefore,

$$\begin{aligned}\cosh c_k &< \frac{\cosh a_k}{\cosh b_k} \leq \frac{\cosh(r_k+1)}{\cosh(r_{k-1})} < \frac{\cosh(r_k)(\cosh(1)+\sinh(1))}{\cosh(r_{k-1})} \\ &= \frac{\sin(\varepsilon_{k-1}/2)(\cosh(1)+\sin(1))}{\sin(\varepsilon_k/2)} \quad \text{(by using (2.3))} \\ &= \frac{\sin(\varepsilon_{k-1}/2)(\cosh(1)+\sinh(1))}{\sin((\alpha\beta)^h\varepsilon_{k-1}/2)} \\ &\leq C_2 \text{ (a constant independent of } k)\end{aligned}$$

We observe that this estimate is independent of the Riemannian angle between the sides a_k and b_k and, in particular, holds when this angle is ε_k.

Now choose s' such that $s' > \max\{r_0, C_1/2, C_2/2\}$. Put

$$\mathcal{U}_{s'}^k = \{\tilde{U} \in \mathcal{U}_{s'} : r_{k-1} \leq (\text{distance of } \tilde{\mathcal{U}} \text{ from } \tilde{p}) < r_k\}.$$

Since the horoballs $\mathcal{U}$ are mutually disjoint, the horoballs in $\mathcal{U}_{s'}$ are at least $2s'$ distance apart from each other. Let $\mathcal{H}$ be any member of the family $\mathcal{U}_{s'}^k$ and let q be the point on $\mathcal{H}$ nearest to $\tilde{p}$. Let q_0 be a point on $\mathcal{H}$ such that the geodesic $\gamma_{\tilde{p}q_0}$ joining $\tilde{p}$ and q_0 is tangential to $\mathcal{H}$ at q_0. If $\tilde{d}(q, q_0)$ denotes the Riemannian distance between q and q_0 in $\tilde{M}$ and $h(q, q_0)$, the horospherical distance, we have (see [**HI**, Proposition 4.7]).

$$\tilde{d}(q, q_0) < h(q, q_0) \leq \frac{1}{\coth s} \leq 1$$

for $s \in (0, \infty)$ where $s = \tilde{d}(\tilde{p}, q_0)$. Therefore, the distance between $\tilde{p}$ and q_0 is at most $(r_k + 1)$. And by the work above it follows that the shadow of the ball B_{kh} chosen by $\mathcal{B}$ at the khth stage intersects at most one member of the family $\mathcal{U}_{s'}^k$.

Step 2. Let $P: \tilde{M}\backslash\{\tilde{p}\} \to S_{\tilde{p}}$ be the map obtained by projecting $\tilde{M}$ onto $S_{\tilde{p}}$ along geodesics emanating from $\tilde{p}$. Then the image $P(\tilde{U})$ of a closed horoball $\tilde{U}$ is diffeomorphic to a closed ball in $S_{\tilde{p}}$ and is a convex subset of $S_{\tilde{p}}$ since $\tilde{U}$ is convex in $\tilde{M}$. Let $s_0 > 0$ be such that for any $k \geq 0$ and $\tilde{U} \in \mathcal{U}_{s'+s_0}$, $P(\tilde{U}) \cap \tilde{\sigma}$ is an interval.

We now find an $s'' \geq s_0$ such that for any horoball $\tilde{U} \in \mathcal{U}^k_{s'+s''}$, $P(\tilde{U}) \in S_{\tilde{p}}$ is contained in a ball of radius less than $\nu\rho_{kh}$. To achieve this, consider a geodesic triangle in $\tilde{M}$ with two sides of length $(r_{k-1}+s)$ (for sufficiently large s) each and the third side of length equal to 1. If τ_k denotes the angle between the sides of length $(r_{k-1}+s)$, we first note that $P(\tilde{U})$ as above is contained in a ball of radius less than τ_k. Then, using the cosine formula A.1(iii) for the above triangle we have,

$$\cos(1) \geq 1 + 2\sinh^2(r_{k-1}+s)\sin^2(\tau_k/2).$$

That is,

$$\begin{aligned}\sin^2(\tau_k/2) &\leq \frac{\cosh(1)-1}{2\sinh^2(r_{k-1}+s)} < \frac{\cosh(1)-1}{2\cosh^2(r_{k-1})\sinh^2 s}\\ &= \frac{(\cosh(1)-1)\sin^2(\varepsilon_{k-1}/2)}{2\sinh^2 s} \quad \text{(by using (2.3))}\\ &= f(s)\sin^2(\varepsilon_{k-1}/2)\end{aligned}$$

where $f(s) = (\cosh(1)-1)/2\sinh^2 s$. The function $f(s)$ decreases to zero as s tends to infinity. Thus choosing a suitable s we can make the right side as small as we like uniformly for all k. In particular, we can choose an s'' such that

$$\tau_k < (\nu(\alpha\beta)^h(\varepsilon_{k-1}/2))/\delta = (\nu(\alpha\beta)^{kh}\rho_0)/\delta = (\nu\rho_{kh})/\delta.$$

This implies that the radius of the interval $P(\tilde{U}) \cap \tilde{\sigma}$ under the metric on $S_{\tilde{p}}$ restricted to $\tilde{\sigma}$ (which we have denoted by d') is less than $(\nu\rho_{kh})/\delta$. Since the distances d and d' are bi-lipschitz related, we conclude that the condition (2.3) holds for the interval $P(\tilde{U}) \cap \tilde{\sigma}$ under the distance d with respect to the ball B_{kh}, where the estimate for τ_k corresponds to the estimate on ρ as in (2.3).

Step 3. (Completion of the proof.) Starting from $\mathcal{B}$'s choice of a closed interval $B_0 \subset \tilde{\sigma}$ of radius ρ_0, we find numbers s' and s'' as indicated in Steps 1 and 2. From the arguments in Step 1 we see that, for any k, the interval B_{kh} chosen by $\mathcal{B}$ at the khth stage of the (α, β)-game intersects $P(\tilde{U})$ for at most one member $\tilde{U}$ of $\mathcal{U}^k_{s'+s''}$. And whenever B_{kh} intersects any such $P(\tilde{U})$, by the estimates of Step 2 and using the Lemma 2.1, $\mathcal{A}$ can make his choices, during the next h steps, in such a way that $B_{(k+1)h}$ does not intersect $P(\tilde{U}) \cap \tilde{\sigma}$. For $k = 0$ this is trivially true since $s' > r_0$. This way, $\mathcal{A}$ can inductively force $\mathcal{B}$ to make his choices such that, for any k, $B_{(k+1)h}$ does not intersect the P images of any of the horoballs $\mathcal{U}^0_{s'+s''}, \ldots, \mathcal{U}^k_{s'+s''}$. This means that the geodesic in the direction of the unique point of intersection of the (α, β)-game, stays away from all the horoballs from $\mathcal{U}_{s'+s''}$.

This proves that the set $\tilde{\sigma} \cap C_{\tilde{p}}(\mathcal{E})$ is an (α, β)-winning set for $\alpha, \beta \in (0, 1)$ such that $1 - 2\alpha + \alpha\beta > 0$; this means, in particular, that it is α-winning for $\alpha \in (0, 1/2)$. Since M has only finitely many ends and since intersections of any countable family of

α-winning sets is α-winning (cf [**S**, Theorem 2]), we get that $\tilde{\sigma} \cap C_{\tilde{p}} = \cap(\tilde{\sigma} \cap C_{\tilde{p}}(\mathcal{E}))$ is an α-winning set. Since Γ acts as a group of isometries on $\tilde{M}$ and $\tilde{\sigma}$ is a lift of σ and since a geodesic in M is bounded if and only if it stays away from a neighborhood of each of the ends, it follows that $\sigma \cap C_p$ is an α-winning set in σ. As mentioned earlier, by a theorem of Schmidt [**S**, Corollary 2 §11] this implies that $\sigma \cap C_p$ has Hausdorff dimension 1.

Appendix. *Trigonometry of rank-1 symmetric spaces*

We consider a rank-1 Riemannian symmetric space of non-compact type. Since three points in such a space always lie in a totally geodesic submanifold isometric to a complex hyperbolic plane it suffices to consider triangles in the latter. Let $(,)$ denote the standard hermitian form on $\mathbb{C}^2$, i.e., $(u, v) = u_1\bar{v_1} + u_2\bar{v_2}$. The complex hyperbolic plane $\mathbb{CH}^2$ is the open unit ball $\mathbb{B} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$ in $\mathbb{C}^2$ together with the distance function d given by

$$\cosh(d(u, v)) = \frac{|1 - (u, v)|}{(1 - \|u\|^2)^{1/2}(1 - \|v\|^2)^{1/2}} \tag{$*$}$$

With this normalization the sectional curvature of $(\mathbb{B}, d)$ is pinched between -4 and -1.

Consider unit tangent vectors $X, Y \in T_O(\mathbb{CH}^2) \cong \mathbb{C}^2$ where O is the origin. Put $\langle X, Y\rangle = \mathrm{Re}(X, Y) = \cos(\lambda)$ and $\langle X, iY\rangle = \mathrm{Re}(X, iY) = \cos(\theta)$. For a triangle ABC with vertex $C = O$ we have three sides $a = d(O, B)$, $b = d(O, A)$, $c = d(A, B)$ and six angles $(\lambda(A), \theta(A))$, $(\lambda(B), \theta(B))$, $(\lambda(C), \theta(C))$. Note that λ is the usual Riemannian angle. Define a third angle μ by $1 = \cos^2\theta + \cos^2\mu + \cos^2\lambda$.

PROPOSITION A.1. (Laws of sines and cosines.)

$$\begin{aligned}
&\text{(i)} & \frac{\cos\mu(A)}{\sinh a} &= \frac{\cos\mu(B)}{\sinh b} = \frac{\cos\mu(C)}{\sinh c}\\
&\text{(ii)} & \frac{\cos\theta(A)}{\sinh(2a)} &= \frac{\cos\theta(B)}{\sinh(2b)} = \frac{\cos\theta(C)}{\sinh(2c)}\\
&\text{(iii)} & \cosh^2 c &= (\cosh a\cosh b - \sinh a \sinh b\cos\lambda(C))^2\\
&&&\quad +(\sinh a\sinh b\cos\theta(C))^2.
\end{aligned}$$

Proof. For (i) and (ii) see [**L**], [**H**] and [**B**]. To prove (iii), write $A = xX$, $B = yY$ with $X, Y \in T_O(\mathbb{CH}^2)$ and $\|X\| = \|Y\| = 1$. Equation $(*)$ yields $\cosh a = \cosh(d(O, B)) = 1/(1 - x^2)^{1/2}$ or $x = \tanh a$; similarly, $y = \tanh b$. Again by $(*)$

$$\cosh^2 c = \cosh^2(d(A, B)) = \frac{|1 - (A, B)|^2}{(1 - \|A\|^2)(1 - \|B\|^2)}$$

and

$$|1 - (A, B)|^2 = |1 - xy(X, Y)|^2 = (1 - xy\cos\lambda(C))^2 + (xy\cos\theta(C))^2$$

Insert this in the formula above to deduce (iii). □

Next consider a triangle in $\mathbb{CH}^2$ with one vertex, say B, at infinity with $d(A, C) = b < \infty$ and with angle $\lambda(C) = \pi/2$. The second angle $\mu(C)$ (or $\theta(C)$) is variable. We calculate the angles at A in terms of $\mu = \mu(C)$ and b.

PROPOSITION A.2. (Angles of parallelism.)

$$\text{(i)} \quad \cos^2 \mu(A) = \frac{\cos^2 \mu}{1 + \sinh^2 b(1 + \sin^2 \mu)}$$

$$\text{(ii)} \quad \cos^2 \theta(A) = \frac{\sin^2 \mu}{(1 + \sinh^2 b(1 + \sin^2 \mu))^2}.$$

Proof. By Proposition A.1(i) we have

$$\cos^2 \mu(A) = \cos^2 \mu \frac{\sinh^2 a}{\sinh^2 c}.$$

Since $\lambda(C) = \pi/2$ and thus $\theta(C) = \pi/2 - \mu(C) = \pi/2 - \mu$, Proposition A.1(iii) yields,

$$\cosh^2 c = \cosh^2 a \cosh^2 b + \sinh^2 a \sinh^2 b \sin^2 \mu$$

and hence

$$\sinh^2 c = \sinh^2 a + \sinh^2 b + \sinh^2 a \sinh^2 b(\sin^2 \mu + 1).$$

Insert this in the equation above for $\cos(\mu(A))$ and then take the limit for $a \to \infty$ to obtain (i).

(ii) follows by similar arguments using Proposition A.1(ii). □

PROPOSITION A.3.

$$\text{(i)} \quad \cos^2 \theta(A, \mu) \leq \frac{1}{\cosh^4 b}$$

$$\text{(ii)} \quad \frac{1}{\cosh(2b)} \leq \sin \lambda(A, \mu) \leq \frac{1}{\cosh b}.$$

Proof. To prove (i), we use Proposition A.2(ii) and see that

$$\cos^2 \theta(A, \mu) = \frac{\sin^2 \mu}{(1 + \sinh^2 b(1 + \sin^2 \mu))^2} \leq \frac{1}{\cosh^4 b}.$$

(ii) follows from [**HI**, Proposition 4.4] with the correction that $\beta_b \leq \beta \leq \beta_a$. This can also be deduced by direct computation using Proposition A.2 and $\sin^2 \lambda(A) = 1 - \cos^2 \lambda(A) = \cos^2 \theta(A) + \cos^2 \mu(A)$. □

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